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COVARIANCE MATRIX ESTIMATOR PERFORMANCE IN NON-GAUSSIAN SPHERICALLY INVARIANT RANDOM PROCESSES

James H. Michels

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13. ABSTRACT (Maximum 200 words)

This report describes the performance of the covariance matrix estimator in non-Gaussian spherically invariant random processes (SIRP). Analytic expressions are derived for the variance of the estimator. Specific consideration is given to the special cases of Weibull and K-distributed processes as a function of the shape parameter. Validation is achieved via Monte-Carlo simulation. The expressions reveal the increase in the estimator variance for non-Gaussian SIRP's as well as the sample support size required to reduce the variance to that of the Gaussian case.

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1.0 Introduction

In many applications, such as the phased array radar problem, we have an observed J channel data vector $\underline{z}(n)$ at time n which may contain a desired signal $\underline{s}(n)$ and an unwanted additive disturbance $\underline{y}(n)$ which may contain additive temporally correlated noise (clutter) $\underline{c}(n)$ and uncorrelated thermal white noise $\underline{w}(n)$. For the detection problem, a decision must be made between the two hypotheses

$$H_0: \underline{z}(n) = \underline{y}(n) \qquad\qquad n=1,2,...,N$$
$$H_1: \underline{z}(n) = \underline{s}(n) + \underline{y}(n) \qquad\qquad n=1,2,...,N. \qquad (1)$$

For high resolution radar, the spatial clutter statistics deviate from the Gaussian model [1,2]. The non-Gaussian nature of the clutter can cause severe performance reduction in applications such as space-time processing and bearing estimation using phased array systems. In these applications, a covariance matrix is used to estimate the unwanted interference noise in the data cell under test. Ensemble averaging is replaced by averaging over available adjacent secondary data cells (i.e., range cells assumed to be free of the desired signal) assuming independence between range cells. This estimator is the sample matrix estimator and we refer to this case as that of a *spatially-averaged* estimate. The presence of the spatially non-Gaussian clutter causes a considerable increase in the variance of the sample covariance matrix estimator as compared to that for the Gaussian case.

In other applications such as the airborne radiometer [3], a *time-averaged* estimate is used by averaging over a time sequence of data. In this paper, the variance of the *spatially-averaged* and *time-averaged* covariance estimators are described analytically in terms of parameters of non-Gaussian models pertinent to these applications.

Two families of distributions for SIRP's have received considerable attention; namely, the K-distribution [1]

$$f_R(r) = \frac{\beta^{\alpha+1} r^\alpha}{2^{\alpha-1}\Gamma(\alpha)} K_{\alpha-1}(\beta r) \qquad r \geq 0 \qquad \beta,\alpha > 0 \qquad (1)$$

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and the Weibull [2],

$$f_R(r) = \beta\alpha r^{\alpha-1}\exp(-\beta r^\alpha) \qquad r \geq 0 \qquad \beta,\alpha > 0 \qquad (2)$$

where $\Gamma(\cdot)$ is the Eulerian function and $K(\cdot)$ is the modified Bessel function of the second kind of order α. The parameters β and α are the scale and shape parameters, respectively. The former is related to the variance of the quadrature components of the clutter, while the latter controls the tails of the distribution.

The complex envelope $y(n,k)$ at time n and range cell k of the compound-Gaussian clutter model [2,4] has been considered; i.e.,

$$y(n,k) = v(n,k)x(n,k) \qquad (3)$$

where the complex Gaussian process $x(n,k)$ (the speckle component) modulates a nonnegative process $v(n,k)$ containing discretes (texture) which is statistically independent of $x(n,k)$. When $v(n,k)$ has a long coherence time, it can be considered to be a random variable with respect to k, but constant over time n so that eq(3) reduces to the case of a spherically invariant random process (SIRP) [5,6]

$$y(n,k) = v(k)x(n,k). \qquad (4)$$

Eq(4) is known as the representation theorem [5] and the probability density function (pdf) $f_v(v)$ is called the characteristic pdf. It controls the pdf of $y(n,k)$. The Weibull and K-distributed processes are both special cases of the SIRP's [6]. We note that conditioned on $v(k)$, the time sequence $y(n)$ is Gaussian. This implies that a sequence of data from a specific range cell is locally Gaussian, but non-Gaussian when observed across cells. Furthermore, $y(n,k)$ is non-ergodic; i.e., averaging over range cells where $v(k)$ is a random variable from cell-to-cell does not provide the same statistic when averaging over time with $v(k)$ constant.

In this paper, the variance of both the *spatially-averaged* and *time-averaged* covariance estimates is derived. The special cases of Weibull and K-distributed processes are noted specifically. Another important contribution of this paper is the use of the representation theorem of eq(4) in the derivation. This theorem enables a significant simplification of the procedure.

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2.0 Derivation

2.1 Spatially Averaged Estimator

We first consider the *spatially-averaged* covariance estimator; i.e. averaging performed over range cells as described above. Consider the J channel vector $\underline{y}(n,k)$ with JxJ covariance matrix $R=E[\underline{y}(n,k)\underline{y}^H(n,k)]$ where H denotes the Hermitian transpose. Suppressing the time dependence, let $\underline{y}(k)$ be a J channel vector of observation processes from the kth sample (or secondary range cell). Consider the JxJ covariance matrix estimate defined as

$$\hat{R} = \frac{1}{K}\sum_{k=1}^{K} \underline{y}(k)\underline{y}^H(k). \tag{5}$$

Assuming statistical independence over k, $\hat{R}$ is the sample covariance matrix, and provides a maximum likelihood estimate for Gaussian processes [7]. The ij^{th} element $\hat{r}_{ij}$ of $\hat{R}$ is defined as

$$\hat{r}_{ij} = \frac{1}{K}\sum_{k=1}^{K} y_i(k)y_j^*(k). \tag{6}$$

The estimator $\hat{r}_{ij}$ is unbiased since $E[\hat{r}_{ij}] = r_{ij}$. Thus the variance and error variance are identical. The variance V_{rs} of $\hat{r}_{ij}$ is defined as,

$$V_{rs} = E\{[\hat{r}_{ij} - E(\hat{r}_{ij})][\hat{r}_{ij} - E(\hat{r}_{ij})]^*\}$$
$$= E(\hat{r}_{ij}\hat{r}_{ij}^*) - E(\hat{r}_{ij})E(\hat{r}_{ij}^*) \tag{7}$$

where the subscript s on V denotes the *spatially -averaged* case. Using (6) in (7) and interchanging the summation and expectation operations,

$$V_{rs} = \frac{1}{K^2}\sum_{m=1}^{K}\sum_{p=1}^{K} E\{y_i(m)y_j^*(m)y_i^*(p)y_j(p)\}$$
$$-\frac{1}{K^2}\sum_{m=1}^{K}\sum_{p=1}^{K} E[y_i(m)y_j^*(m)]E[y_i^*(p)y_j(p)] \tag{8}$$

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where m and p denote the mth and p^{th} range cells, respectively. Using the representation theorem from eq(4) and noting the statistical independence of v and x_i, the first expectation in eq(8) becomes

$$E\{y_i(m)y_j^*(m)y_i^*(p)y_j(p)\} = E[v_m^2 v_p^2]\, E\{x_i(m)x_j^*(m)x_i^*(p)x_j(p)\}. \qquad (9)$$

For zero mean, jointly Gaussian scalar random variables $x_i(\cdot)$, $x_j(\cdot)$, eq(9) reduces to

$$
\begin{aligned}
E\{y_i(m)y_j^*(m)y_i^*(p)y_j(p)\} = E[v_m^2 v_p^2] \big\{ & E[x_i(m)x_j^*(m)]E[x_i^*(p)x_j(p)] \\
& + E[x_i(m)x_i^*(p)]E[x_j^*(m)x_j(p)] \big\} \\
& + E[x_i(m)x_j(p)]E[x_j^*(m)x_i^*(p)].
\end{aligned}
\qquad (10)
$$

Furthermore, when these processes are complex, circularly symmetric, the expectations in the last term of eq(10) are zero, so that

$$
\begin{aligned}
E\{y_i(m)y_j^*(m)y_i^*(p)y_j(p)\} = E[v_m^2 v_p^2] \big\{ & E[x_i(m)x_j^*(m)]E[x_i^*(p)x_j(p)] \\
& + E[x_i(m)x_i^*(p)]E[x_j^*(m)x_j(p)] \big\}.
\end{aligned}
\qquad (11)
$$

Assuming spatial stationarity over range cells, eq(11) becomes

$$E\{y_i(m)y_j^*(m)y_i^*(p)y_j(p)\} = E[v_m^2 v_p^2]\, \{\, |r_{ij}^x(0)|^2 + r_{ii}^x(m-p)[r_{jj}^x(m-p)]^* \,\} \qquad (12)$$

where the superscript x denotes that the correlation functions describe the Gaussian processes defined in eq(4). Similarly, the expectations in the second summation term of eq(8) become

$$E[y_i(m)y_j^*(m)] = E[v_m^2]\, r_{ij}^x(0) \qquad (13a)$$

and

$$E[y_i^*(p)y_j(p)] = E[v_p^2]\, [r_{ij}^x(0)]^*. \qquad (13b)$$

Using eqs(12) and (13) in (8),

$$V_{rs} = \frac{1}{K^2} \sum_{m=1}^{K} \sum_{p=1}^{K} E[v_m^2 v_p^2] \left\{ |r_{ij}^x(0)|^2 + r_{ii}^x(m\text{-}p)[r_{jj}^x(m\text{-}p)]^* \right\}$$

$$- \frac{1}{K^2} \sum_{m=1}^{K} \sum_{p=1}^{K} E[v_m^2] E[v_p^2] |r_{ij}^x(0)|^2$$

$$= \frac{1}{K^2} \sum_{m=1}^{K} \sum_{p=1}^{K} \left\{ E[v_m^2 v_p^2] r_{ii}^x(m\text{-}p)[r_{jj}^x(m\text{-}p)]^* + D |r_{ij}^x(0)|^2 \right\} \tag{14}$$

where

$$D = E[v_m^2 v_p^2] - E[v_m^2] E[v_p^2] \qquad m \neq p \tag{15a}$$
$$= E[v^4] - \left[E[v^2] \right]^2 \qquad m = p. \tag{15b}$$

Defining q=m-p, eq(14) becomes

$$V_{rs} = \frac{1}{K^2} \sum_{q=-(K-1)}^{K-1} [K\text{-}|q|] \left\{ E[v_m^2 v_{m-q}^2] r_{ii}^x(q)[r_{jj}^x(q)]^* + D |r_{ij}^x(0)|^2 \right\} \quad m = 1,2,...,K. \tag{16}$$

Eq(16) can be rewritten as

$$V_r = \frac{1}{K} \left\{ E[v^4] r_{ii}^x(0)[r_{jj}^x(0)]^* + \left\{ E[v^4] - E^2[v^2] \right\} |r_{ij}^x(0)|^2 \right\}$$

$$+ \frac{1}{K} \sum_{\substack{q=-(K-1) \\ q \neq 0}}^{K-1} \left[1 - \frac{|q|}{K} \right] \left\{ E[v_m^2 v_{m-q}^2] r_{ii}^x(q)[r_{jj}^x(q)]^* + D |r_{ij}^x(0)|^2 \right\}$$

$$m = 1,2,...,K \tag{17}$$

where the first term is the q=0 (i.e. the m=p) term. Eq(17) is a general expression for the variance of the sample matrix estimator in non-Gaussian SIRP disturbance. We note that the summation term involves the correlation of the disturbance processes over the range cells.

In the special case where v is statistically independent over range cells,

$$E[v_m^2 v_{m-q}^2] = E[v_m^2] E[v_{m-q}^2] \qquad m = 1,2,...,K \qquad (q \neq 0). \tag{18}$$

From eq(15a), D=0 for this case. Further, if the Gaussian processes $x_i(k)$ are uncorrelated (and therefore statistically independent) over range cells,

$$r_{bb}^{x}(q) = \begin{cases} \sigma_x^2 & q=0 \\ 0 & q \neq 0 \end{cases} \qquad \text{for } b = i,j. \qquad (19)$$

where σ_x^2 denotes the channel variance associated with $x_i(k)$ and will be assumed equal on all channels for simplicity. When eq(19) holds, all terms in the summation of eq(17) are zero, so that

$$V_{rs} = \frac{1}{K} \left\{ E[v^4]\sigma_x^4 + \left\{ E[v^4] - E[v^2]^2 \right\} |r_{ij}^x(0)|^2 \right\}. \qquad (20)$$

We note that the quantities σ_x^4 and $r_{ij}^x(0)$ relate to the statistics of the processes $x(k)$ which are not directly measurable. However, recognizing that $\sigma_y^4 = E[v^2]^2 \sigma_x^4$ and $|r_{ij}^y(0)|^2 = E[v^2]^2 |r_{ij}^x(0)|^2$, we can easily express eq(20) in terms of the statistics of the observed data processes $y(k)$ with the result that

$$V_{rs} = \frac{1}{K} \frac{1}{E[v^2]^2} \left\{ E[v^4]\sigma_y^4 + \left\{ E[v^4] - E[v^2]^2 \right\} |r_{ij}^y(0)|^2 \right\}. \qquad (21)$$

We note that the variance of this estimator converges to zero as K approaches infinity. Thus, the sample matrix (using *spatial* averaging) is a consistent estimator for non-Gaussian SIRP's. However, this quantity is now dependent upon the second and fourth moments of v.

We now consider two special cases of the non-Gaussian SIRP models; namely, the Weibull and the K-distribution. For K distributed processes with shape parameter α and scale parameter b, $E[v^4]=\alpha(\alpha+1)b^4$ and $E[v^2]=\alpha/b^2$ (see Appendix A). We let $E[v^2]=1$ without loss of generality so that $E[v^4]= 1+1/\alpha$. For simplicity, consider a diagonal element term [i.e., i=j so that $r_{ij}^x(0)=r_{ii}^x(0)=\sigma_x^2$]. In this case eq(20) becomes

$$V_{rs} = \frac{1}{K}\left[1 + \frac{2}{\alpha}\right]\sigma_x^4 \qquad\qquad i=j \qquad\qquad (22)$$

For Gaussian processes, $\alpha \to \infty$ so that eq(22) reduces to the well known result σ_x^4/K. However, for K-distributed processes with high tails ($\alpha<1$), the $1/\alpha$ term causes a significant increase in the variance of the estimator. For example, with $\alpha=0.5$, V_{rs} is five times larger than the corresponding Gaussian case. For Weibull processes with $E[v^2]=1$ (see Appendix B),

$$E[v^4] = \left(\frac{\alpha}{2}\right)\left[\frac{\Gamma(4/\alpha)}{\Gamma^2(2/\alpha)}\right] \qquad\qquad 0 < \alpha \leq 2$$

and eq(20) becomes

$$V_{rs} = \frac{1}{K}\left\{\alpha\left[\frac{\Gamma(4/\alpha)}{\Gamma^2(2/\alpha)}\right]-1\right\}\sigma_x^4. \qquad\qquad i=j \qquad\qquad (23)$$

For the case of Weibull processes with Rayleigh amplitudes, $\alpha=2$ and eq(23) also reduces to σ_x^4/K. For Weibull processes with high tails (i.e. low α values), V_{rs} increases rapidly with decreasing α. Finally, we note that V_{rs} is inversely related to the beta function $\beta(n,m)$ with $m=n=2/\alpha$ where $\beta(n,m)=\Gamma(m)\Gamma(n)/\Gamma(m+n)$.

2.2 Time-Averaged Estimator

We now consider the *time-averaged* covariance estimator; i.e. averaging performed over N time samples. In this case,

$$\hat{R}(\delta) = \frac{1}{N}\sum_{n=1}^{N-|\delta|} \underline{y}(n)\underline{y}^H(n-\delta) \qquad\qquad (24)$$

where δ denotes temporal lag. Eq(8) is now rewritten as

$$V_{rt} = \frac{1}{N^2}\sum_{n=1}^{N-|\delta|}\sum_{t=1}^{N-|\delta|} E\left\{y_i(n)y_j^*(n-\delta)y_i^*(t)y_j(t-\delta)\right\}$$

$$-\frac{1}{N^2}\sum_{n=1}^{N-|\delta|}\sum_{t=1}^{N-|\delta|} E[y_i(n)y_j^*(n-\delta)]E[y_i^*(t)y_j(t-\delta)] \qquad\qquad (25)$$

where n and t denote the nth and t^{th} time samples, respectively, and the subscript t on V denotes the *time-averaged* case. Again, using the representation theorem from eq(4), but now noting that v(k) has no temporal dependence, the expectations in eq(25) become

$$E\{y_i(n)y_j^*(n-\delta)y_i^*(t)y_j(t-\delta)\} = E[v^4]\, E\{x_i(n)x_j^*(n-\delta)x_i^*(t)x_j(t-\delta)\}. \qquad (26a)$$

$$E[y_i(n)y_j^*(n-\delta)] = E[v^2]E[x_i(n)x_j^*(n-\delta)] \qquad (26b)$$

and

$$E[y_i^*(t)y_j(t-\delta)] = E[v^2]\, E[x_i^*(t)x_j(t-\delta)]. \qquad (26c)$$

Again, for zero mean, jointly Gaussian scalar random variables $x_i(\bullet)$, $x_j(\bullet)$, and assuming temporal stationarity, it follows that

$$V_{rt} = \frac{1}{N^2}\sum_{n=1}^{N-|\delta|}\sum_{t=1}^{N-|\delta|} E[v^4]\left\{|r_{ij}^x(\delta)|^2 + r_{ii}^x(n-t)[r_{jj}^x(n-t)]^*\right\}$$

$$- \frac{1}{N^2}\sum_{n=1}^{N-|\delta|}\sum_{t=1}^{N-|\delta|} E[v^2]^2\, |r_{ij}^x(\delta)|^2$$

$$= \frac{1}{N}\sum_{v=-(N-|\delta|)}^{N-|\delta|}\left[1 - \frac{|\delta|+|v|}{N}\right]\left[E[v^4]r_{ii}^x(v)[r_{jj}^x(v)]^*\right] + \left\{E[v^4] - E[v^2]^2\right\}|r_{ij}^x(\delta)|^2. \qquad (27)$$

where $v = n-t$ and we have used the fact that $|r_{ij}^x(\delta)|^2$ is independent of v. We emphasize that the second term in eq(27) is independent of N. Thus, the variance of the time-averaged estimator does not converge to zero for increasing N which implies that this estimator lacks consistency. This result can be explained by noting the non-ergodic nature of the SIRP clutter process described in section I.

In the special case for i=j and $\delta=0$, V_{rt} expresses the variance $Var(\hat{\sigma}^2)$ of the estimated power $\hat{\sigma}^2$ on a given channel as [8]

$$Var(\hat{\sigma}^2) = \frac{1}{N}\sum_{v=-N}^{N}\left[1 - \frac{|v|}{N}\right]E[v^4]|r_{ii}^x(v)|^2 + \left\{E[v^4] - E[v^2]^2\right\}\sigma_x^4 \qquad (28)$$

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Furthermore, for white noise processes, only the $\nu=0$ term in the summation remains, so that with $r_{ii}^x(0)=\sigma_x^2$

$$\text{Var}(\hat{\sigma}^2) = \frac{1}{N} E[v^4]\sigma_x^4 + \left\{ E[v^4] - E[v^2]^2 \right\} \sigma_x^4 \tag{29}$$

which compares to the result presented in [3]. Again, we note the inconsistency of the estimator. Using the previously defined expressions of $E[v^2]$ and $E[v^4]$, for K-distributed processes

$$\text{Var}(\hat{\sigma}^2) = \frac{1}{N}\left[1 + \frac{1}{\alpha}\right]\sigma_x^4 + \frac{1}{\alpha}\sigma_x^4 \tag{30}$$

while for Weibull processes

$$\text{Var}(\hat{\sigma}^2) = \frac{1}{N}\left(\frac{\alpha}{2}\right)\left[\frac{\Gamma(4/\alpha)}{\Gamma^2(2/\alpha)}\right]\sigma_x^4 + \left\{\left(\frac{\alpha}{2}\right)\left[\frac{\Gamma(4/\alpha)}{\Gamma^2(2/\alpha)}\right] - 1\right\}\sigma_x^4 \tag{31}$$

For the Gaussian case, $\alpha=\infty$ and $\alpha=2$ in eqs(30) and (31), respectively, and $\text{Var}(\hat{\sigma}^2)$ reduces to the known result σ_x^4/N.

3.0 Simulated Results

Validation of eqs(22) and (23) was obtained via Monte-Carlo using 1000 statistically independent realizations. The normalized term KV_r/σ_x^4 is plotted as a function of the shape parameter α for the K-distributed noise and the Weibull noise in Figures 1 and 2, respectively. These curves approach unity (the Gaussian case) as $\alpha \rightarrow \infty$ in Figure 1 for K-distributed noise and at $\alpha=2$ (Rayleigh amplitudes) in Figure 2 for Weibull noise. These curves show the dramatic increase in the variance as a function of the shape parameter. In Figure 3, V_r/σ_x^4 is plotted as a function of K, the number of independent range cell samples, for the Gaussian and K-distributed ($\alpha=0.1$ and 0.5) cases. These curves show that V_r/σ_x^4 decreases linearly with increasing K. Thus, the estimator is consistent when averaging over range cells. This is unlike the time-averaged covariance estimator in which V_r/σ_x^4 converges to a bias level with increasing number of time samples due to the non-ergodic nature of the SIRP's. Finally, they reveal the number of samples K required to reduce the variance of the estimator for the non-Gaussian processes to that of the Gaussian case.

For the time-averaged estimator, we plot the normalized variance $Var(\hat{\sigma}^2)/\sigma_x^4$ versus N for eqs(30) and (31) in Figures 4 and 5, respectively. These figures reveal the significance of the inconsistent time-averaged covariance estimator in non-Gaussian SIRP's.
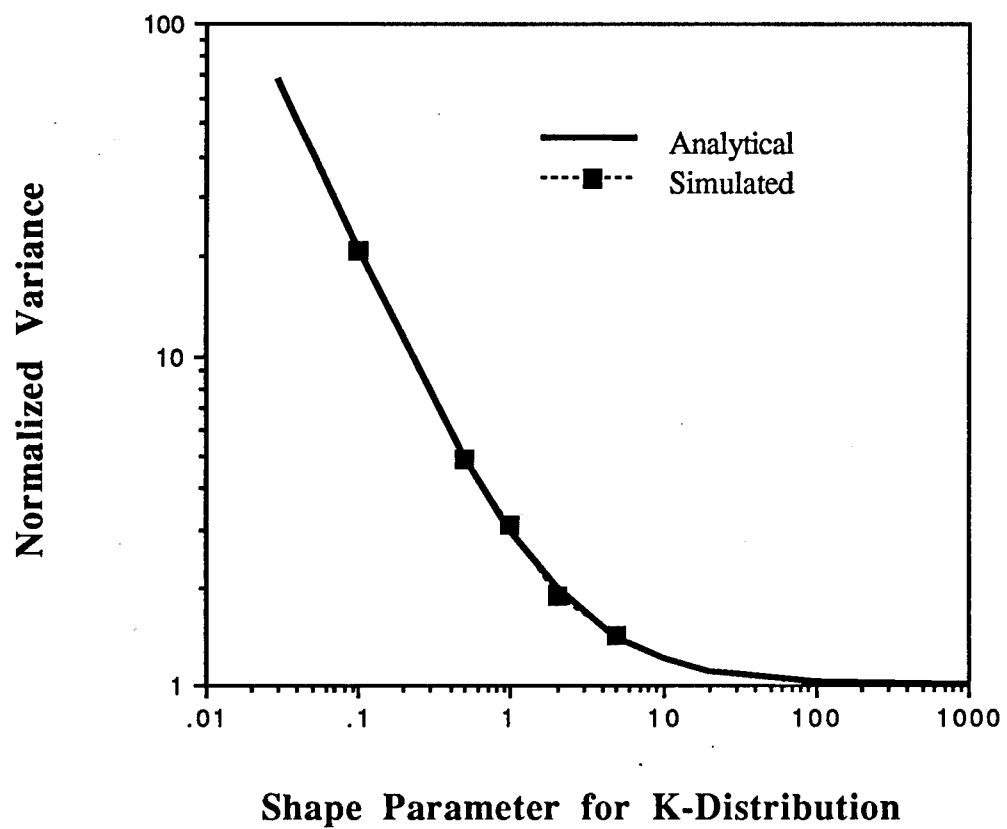
Figure 1 Normalized variance $K(V_r/\sigma_x^2)$ for a diagonal element of the covariance matrix estimator versus the shape parameter for K-distributed noise.
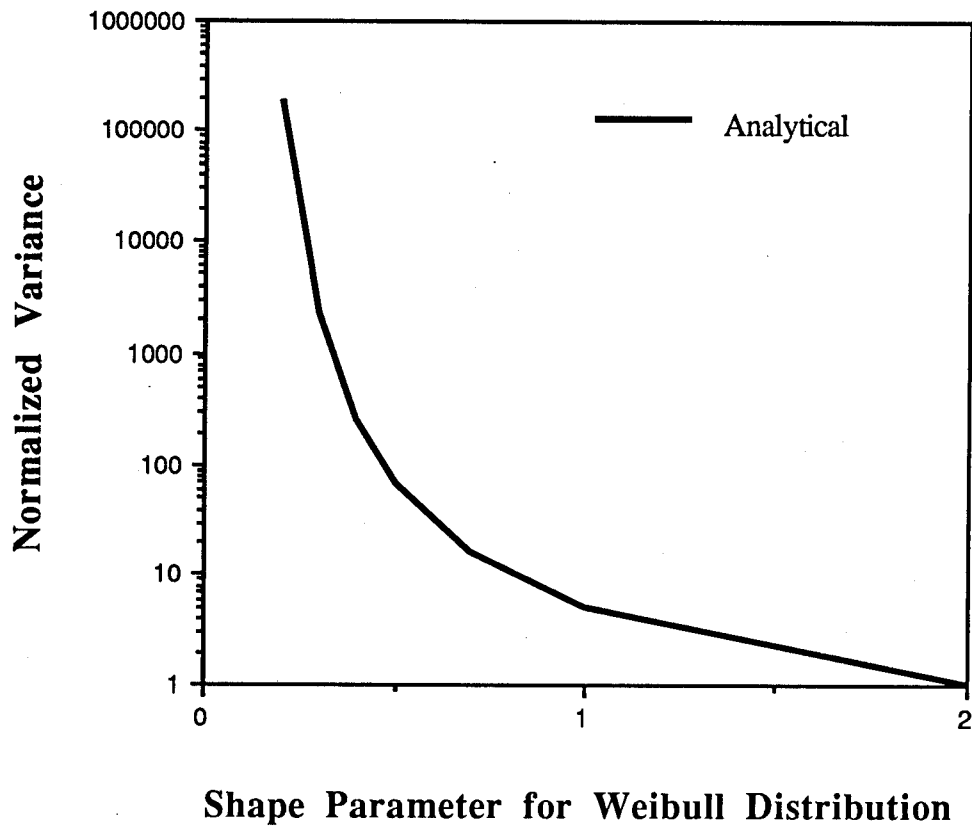
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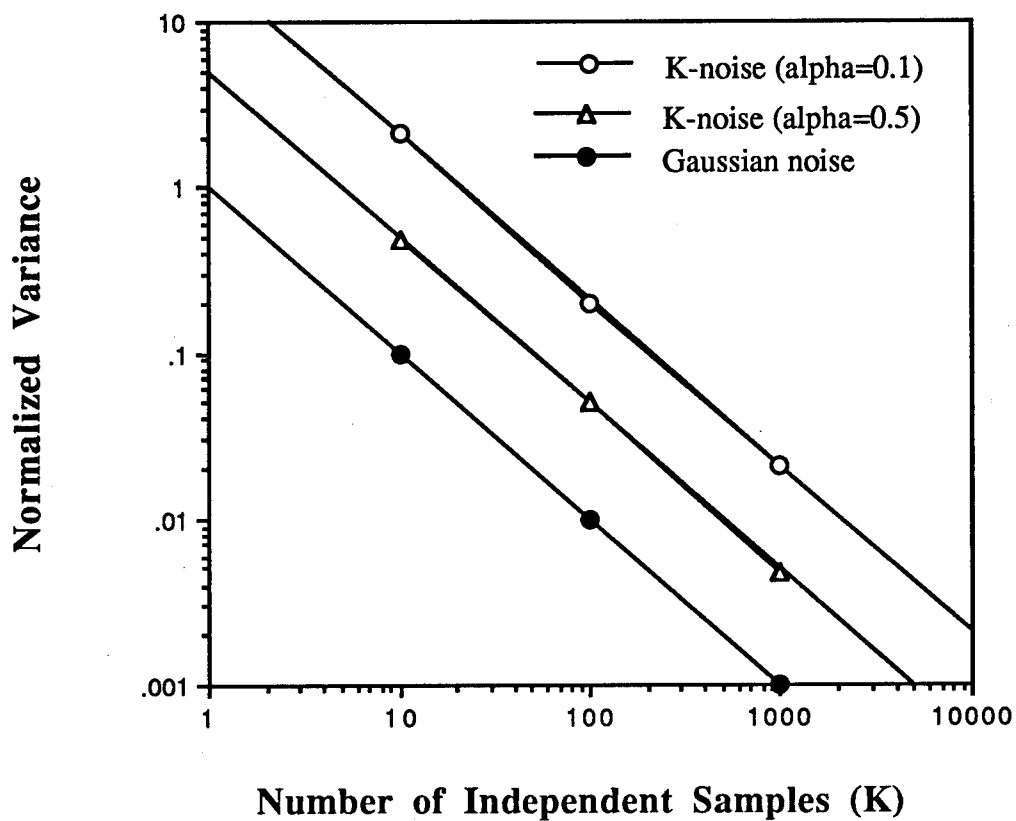
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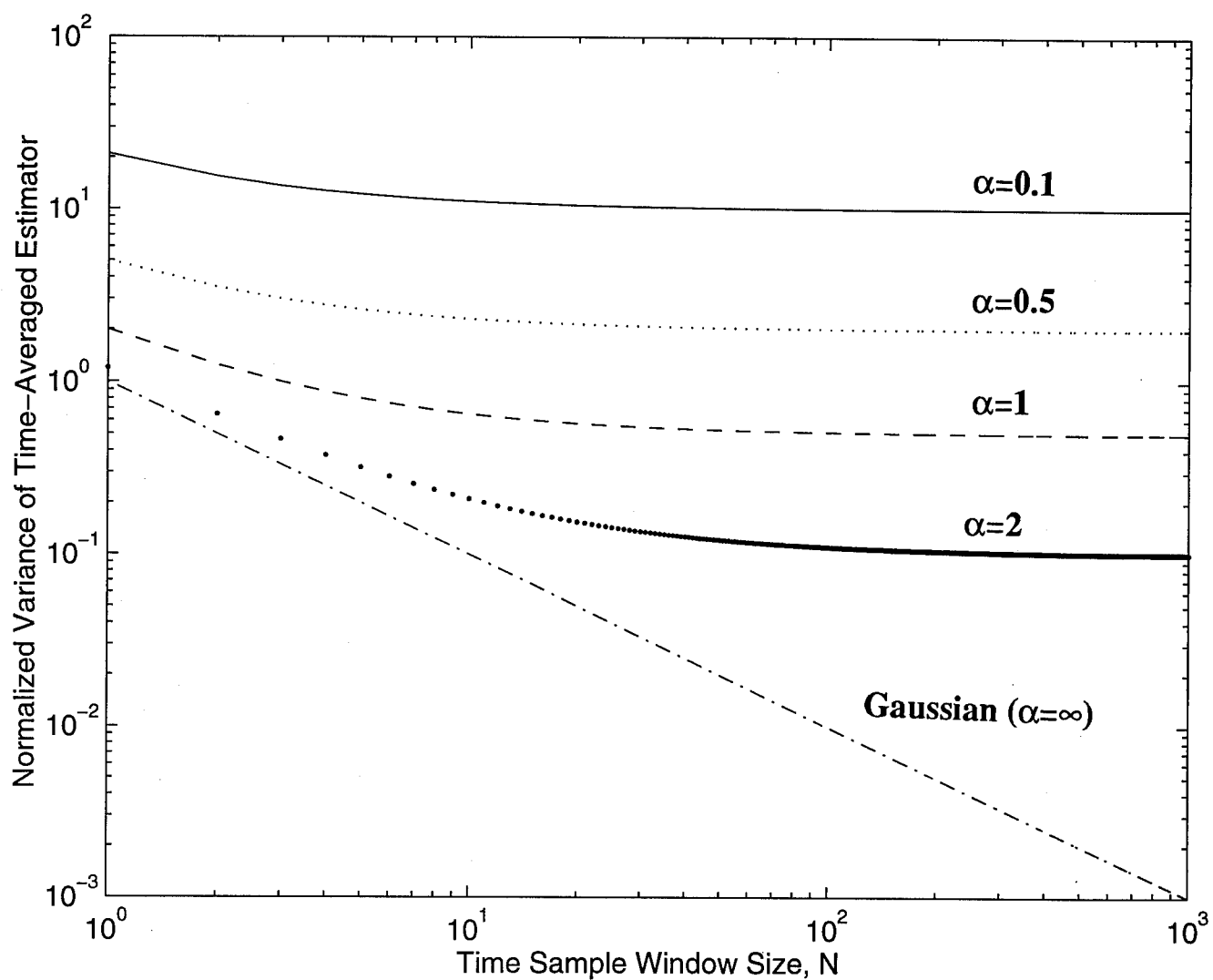
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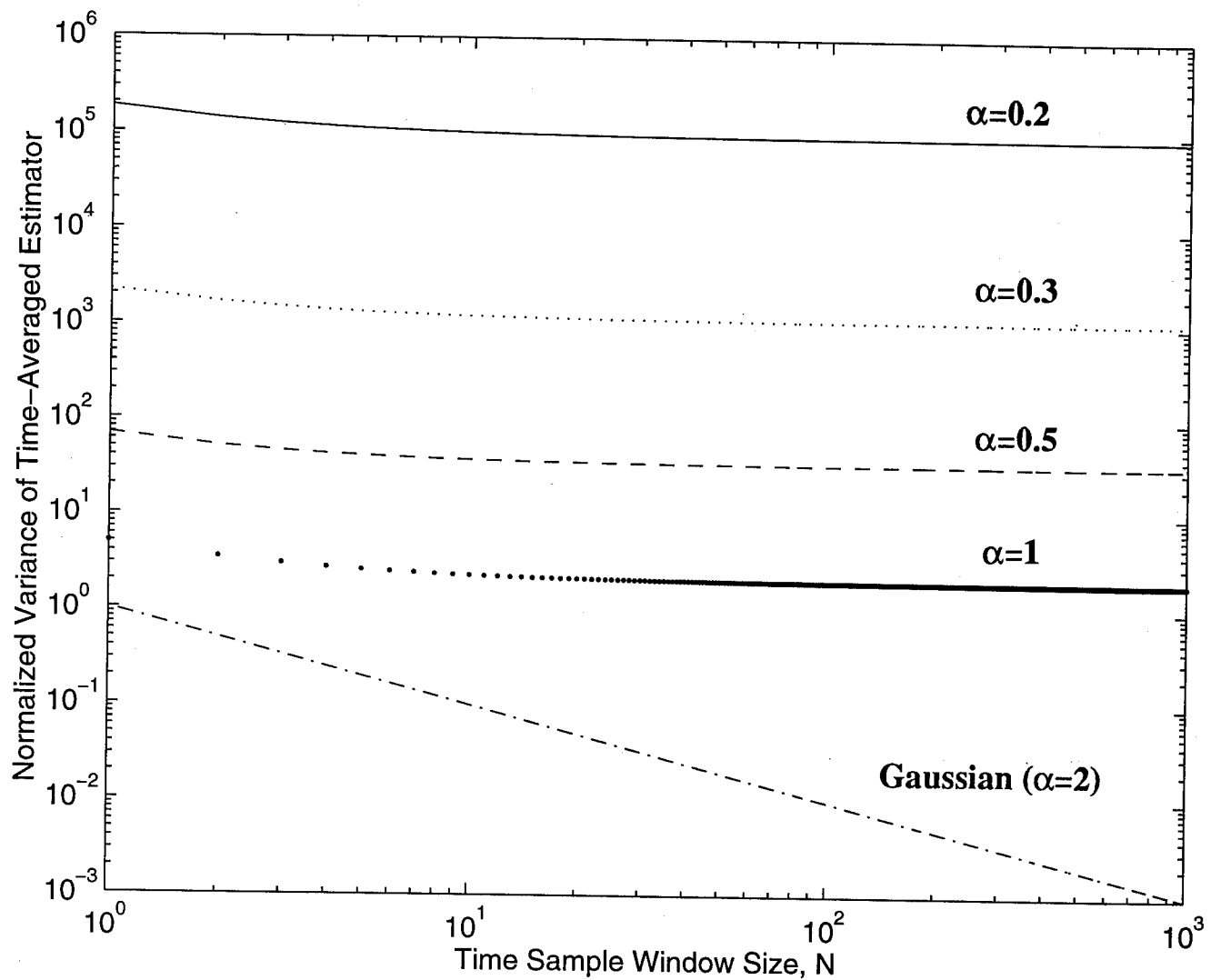
Figure 5 Normalized variance for the time-averaged correlation matrix estimator element for Weibull distributed processes versus the time sample window size N with α as a parameter.

4.0 Summary

This paper presents analytical expressions for the variance of the the estimated covariance matrix elements when the observation data processes are non-Gaussian spherically invariant random processes (SIRP's). Both the *spatially-averaged* and *time-averaged* are considered. These expressions show the significant increase in the variance of the estimates as compared to the Gaussian case. For spatial averaging, they indicate the increase in the number of samples over which averaging must be performed to acheive the same level of performance as the Gaussian case. For the time averaging, the non-ergodic nature of the SIRP clutter precludes convergence of the estimator variance. Validation is acheived through Monte-Carlo simulation.

REFERENCES

[1]   Watts, S. (Ed.)
          Special issue on radar clutter and multipath propagation.
          IEE Proceedings, Pt. F, 138 , 2 (April 1991).

[2]   Sekine, M. and Mao, Y. (1990)
          Weibull radar clutter.
          IEE Radar, Sonar, Navigation and Avionics, Series 3, 1990.

[3]   Sousa, M., and Betz, J.
          Limitations of radiometer performance in spherically invariant noise
          IEEE Transactions on Aerospace and Electronic Systems, Vol. 31,
          no.1, January 1995.

[4]   Conte, E. and Ricci, G.
          Performance prediction in Compound-Gaussian Clutter
          IEEE Transactions on Aerospace and Electronic Systems, Vol. 30,
          No. 2,April 1994.

[5]   Yao, K.
          A representation theorem and its application to spherically invariant
          random processes
          IEEE Transactions on Information Theory, vol. IT-19, pp. 600-608,
          1973.

[6]   Rangaswamy, M.,
          Spherically invariant random processes for radar clutter modeling,
          simulation and distributed identification
          PhD dissertation, Syracuse University, 1992.

[7]   Brennen, L., Reed I.
          Theory of Adaptive Radar
          IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-
          9, pp. 237-252, 1973.

[8]   Michels, J.
          Correlation function estimator performance in non-Gaussian
          spherically invariant random processes
          RL-TR-93-179, Rome Laboratory In-House Report, October, 1993.

APPENDIX A

DERIVATION OF THE EXPRESSION FOR E[v⁴] USING K-DISTRIBUTED PROCESSES

In this Appendix, an analytic expression for $E[v^4]$ is derived in terms of the shape and scale parameters of $f_V(v)$ where $f_V(v)$ is a Gamma distribution and the random variable v is real and non-negative. In this case, the observation data vector process $y(n,k)=v(k)x(n,k)$ described by eq(4) has a K-distribution. For the Gamma distributed $f_V(v)$ with non-negative v,

$$f_V(v) = \frac{2b}{\Gamma(\alpha)} (bv)^{2\alpha-1} \exp(-b^2v^2)$$

so that

$$E[v^4] = \frac{2b}{\Gamma(\alpha)} \int_0^\infty (bv)^{2\alpha-1} v^4 \exp(-b^2v^2)dv$$

$$= \frac{2b}{\Gamma(\alpha)} \int_0^\infty (b)^{2\alpha-1} v^{2\alpha+3} \exp(-b^2v^2)dv. \qquad (A.1)$$

Consider $v=\sqrt{w}/b$ so that $dv/dw=1/2b\sqrt{w}$. Now,

$$f_W(w) = \frac{w^{\alpha-1}\exp(-w)}{\Gamma(\alpha)} \qquad (A.2)$$

and

$$E[v^4] = E[w^2]/b^4. \qquad (A.3)$$

However,

$$E[w^2] = \int_0^\infty \frac{w^{\alpha-1}\exp(-w)}{\Gamma(\alpha)} dw = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1). \qquad (A.4)$$

Using (A.4) in (A.3)

$$E[v^4] = E[w^2]/b^4 = \frac{\alpha(\alpha+1)}{b^4}. \qquad (A.5)$$

APPENDIX B

DERIVATION OF THE EXPRESSION FOR E[s⁴] USING
WEIBULL DISTRIBUTED PROCESSES

In this Appendix, an analytic expression for $E[v^4]$ is derived in terms of the shape and scale parameters for processes with a Weibull distributed amplitude. The Weibull probability density function (pdf) of the amplitude r denoted as f(r) with shape parameter α and scale parameter b is expressed as

$$f(r) = b\alpha \, r^{\alpha-1}\exp(-br^\alpha) \quad r\geq0, \quad 0<\alpha\leq2. \tag{B.1}$$

The conditional pdf f(r|s) is

$$f(r|v) = \frac{r}{v^2}\exp(-r^2/2v^2) \quad r\geq0. \tag{B.2}$$

Using (B.1), the kth moment of r is

$$E[r^k] = \int_0^\infty r^k f(r)dr$$

$$= b\alpha \int_0^\infty r^{k+\alpha-1}\exp(-br^\alpha)dr$$

$$= \left(\frac{1}{b}\right)^{k/\alpha}\Gamma(1+k/\alpha) \tag{B.3}$$

while the kth moment of r conditioned on v is obtained using (B.2), so that

$$E[r^k|v] = \int_0^\infty r^k f(r|v)dr = \int_0^\infty \frac{r^{k+1}}{v^2}\exp(-r^2/2v^2)dr. \tag{B.4}$$

Let $w=r^2/2v^2$, so that $dw= rdr/v^2$ and $r=\sqrt{2}vw^{1/2}$. Eq(B.4) now becomes

$$E[r^k|v] = 2^{k/2} \, v^k \int_0^\infty w^{k/2} \exp(-w)dw$$

$$= 2^{k/2} \, v^k \, \Gamma(1+k/2). \tag{B.5}$$

Now,

$$E[r^k] = \int_0^\infty E[r^k|v] \, f(v)dv$$

$$= 2^{k/2} \, \Gamma(1+k/2) \int_0^\infty v^k \, f(v)dv$$

$$= 2^{k/2} \, \Gamma(1+k/2) \, E[v^k] \tag{B.6}$$

Using (B.3) in (B.6),

$$E[v^k] = \frac{\left(\frac{1}{b}\right)^{k/\alpha} \Gamma(1+k/\alpha)}{2^{k/2} \, \Gamma(1+k/2)} \tag{B.7}$$

Therefore,

$$E[v^4] = \left(\frac{1}{b}\right)^{4/\alpha} \left(\frac{1}{2\alpha}\right) \Gamma(4/\alpha) \tag{B.8}$$

and

$$E[v^2] = \left(\frac{1}{b}\right)^{2/\alpha} \left(\frac{1}{\alpha}\right) \Gamma(2/\alpha). \tag{B.9}$$

For $E[v^2]=1$,

$$\left(\frac{1}{b}\right) = \left[\frac{\alpha}{\Gamma(2/\alpha)}\right]^{\alpha/2}$$

so that

$$E[v^4] = \left(\frac{\alpha}{2}\right) \frac{\Gamma(4/\alpha)}{\Gamma^2(2/\alpha)}. \tag{B.10}$$